

Nonantagonistic noisy duels of discrete type with an arbitrary number of actions

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Abstract

We study a nonzero-sum game of two players which is a generalization of the antagonistic noisy duel of discrete type. The game is considered from the point of view of various criterions of optimality. We prove existence of ε -equilibrium situations and show that the ε -equilibrium strategies that we have found are ε -maxmin. Conditions under which the equilibrium plays are Pareto-optimal are given.

Keywords: noisy duel, payoff function, strategy, equilibrium situation, Pareto optimality, the value of a game.

1 Introduction

The classical duel is a zero-sum game of two players of the following type. The players have certain resources and use them during a given time interval with the goal of achieving success. Use of the resource γ at the moment t leads to success with the probability depending on the amount of resource γ and the time t only (it is usually assumed that the probability of success increases with time). As soon as one player achieves the goal he receives his profit, which is equal to his opponent's loss, and the game ends. Various assumptions about the ways players use their resources and about players receiving information about the opponent's behavior during the game define various kinds of duels [1, 2]. Models were considered where players' resources were discrete (*discrete firing* duels), infinitely divisible (*continuous firing* duels),

continuous for one of the players and discrete for the other one (*mixed* duels, or *fighter-bomber duels*) [5, 6]. Researchers studied *noisy* duels [3, 5, 7], where every player at a given moment of time had complete information about his opponent's behavior up to that moment, and *silent* duels, where no such information was available. At present time duels are considered as classical models of competition [1, 8]. However, their application as such is somewhat limited by the assumption that the players' interests are strictly opposite to each other. Games of the duel type with nonzero sum belong to an unexplored class of infinite games with nonopposite interests.

A nonzero-sum game which is a generalization of the classical antagonistic duel was first considered in [9]; it was a nonantagonistic noisy fighter-bomber duel. Then nonzero-sum duels were studied in [10], [11], [12].

In this paper we study a nonzero-sum game of two players which is a generalization of the antagonistic noisy duel with discrete firing [3, 4]. This paper is an extended version of [13] with complete proofs. It is also a sequel to [11].

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2 Preliminaries from Game Theory

A *game of two players* is a quadruple

$$\Gamma = \{X, Y, K_1(x, y), K_2(x, y)\},$$

where X and Y are sets of the players' *strategies* and $K_j(x, y)$, $j = 1, 2$ are the players' *payoff functions*, which are defined on the Cartesian product $X \times Y$ and determine the j -th player's payoff when Player I uses a strategy $x \in X$ and Player II uses a strategy $y \in Y$. A game Γ is called a *zero-sum game of two players*, or an *antagonistic game* if $K_1(x, y) + K_2(x, y) = 0$ for all $x \in X$, $y \in Y$ and a *nonzero-sum game of two players*, or a *nonantagonistic game* otherwise. The *mixed extension* of a game Γ is the game $\bar{\Gamma} = (\Phi, \Psi, \bar{K}(\varphi, \psi))$, where Φ and Ψ are the sets of distributions on X and Y and $\bar{K}_j(\varphi, \psi)$ ($j = 1, 2$) are the mean values of the payoff functions $K_j(x, y)$ over the distributions $\varphi \in \Phi$, $\psi \in \Psi$. *Pure strategies* of the game $\bar{\Gamma}$ are distributions concentrated in one point $x \in X$ or $y \in Y$, respectively. All the other distributions $\varphi \in \Phi$, $\psi \in \Psi$ are called *mixed strategies*.

A *situation* (x, y) is a pair of the player's strategies. A situation (x_e, y_e) is called an *equilibrium situation* if for all $x \in X$, $y \in Y$ the following inequalities hold:

$$K_1(x, y_e) \leq K_1(x_e, y_e); \quad K_2(x_e, y) \leq K_2(x_e, y_e). \quad (1)$$

The vector (v^1, v^2) , where $v^1 = K_1(x_e, y_e)$ and $v^2 = K_2(x_e, y_e)$, is called the *equilibrium value* corresponding to the equilibrium situation (x_e, y_e) , and a strategy included in some equilibrium situation is called an *equilibrium strategy*.

An infinite game with nonantagonistic interests may not admit an equilibrium situation. A situation $(x^\varepsilon, y^\varepsilon)$ is called an ε -*equilibrium situation* if for any $x \in X$, $y \in Y$ the following inequalities hold:

$$K_1(x, y^\varepsilon) - \varepsilon \leq K_1(x^\varepsilon, y^\varepsilon); \quad K_2(x^\varepsilon, y) - \varepsilon \leq K_2(x^\varepsilon, y^\varepsilon). \quad (2)$$

A player's strategy included in some ε -equilibrium situation will be called an ε -*equilibrium strategy*. If the limits

$$v^j = \lim_{\varepsilon \rightarrow 0} K_j(x^\varepsilon, y^\varepsilon) \quad (j = 1, 2),$$

exist, the vector (v^1, v^2) will be called the *equilibrium value* corresponding to a set of ε -equilibrium situations $\{(x^\varepsilon, y^\varepsilon)\}$.

A strategy $x_m \in X$ is called a *maxmin* strategy of Player I if the function $m_1(x) = \inf_{y \in Y} K_1(x, y)$ attains its maximal value in it. Analogously, a strategy $y_m \in Y$ is called a *maxmin* strategy of Player II if the function $m_2(y) = \inf_{x \in X} K_2(x, y)$ attains its maximal value in it. The vector

$$w = (w^1, w^2), \text{ where } w^1 = \max_{x \in X} \inf_{y \in Y} K_1(x, y), \quad w^2 = \max_{y \in Y} \inf_{x \in X} K_2(x, y),$$

is called the *maxmin value* of a nonzero-sum game. The value w^j is the best guaranteed payoff of the j -th player.

In an infinite game maxmin strategies may not exist. In this case ε -*maxmin* strategies are considered. A strategy $x_m^\varepsilon \in X$ is called an ε -*maxmin* strategy of Player I if

$$m_1(x_m^\varepsilon) > w^1 - \varepsilon, \text{ where } w^1 = \sup_{x \in X} \inf_{y \in Y} K_1(x, y),$$

and w^1 is called the maxmin value of the game of Player I. Analogously one defines an ε -maxmin strategy of Player II: a strategy $y_m^\varepsilon \in X$ is called an ε -maxmin strategy of Player II if

$$m_2(y_m^\varepsilon) > w^2 - \varepsilon, \text{ where } w^2 = \sup_{y \in Y} \inf_{x \in X} K_2(x, y),$$

and w^2 is called the maxmin value of the game of Player II.

If an antagonistic game admits equilibrium situations, then the equilibrium values corresponding to them coincide and are equal to the maxmin value. Equilibrium strategies of an antagonistic game are maxmin and they are called *optimal*, while ε -equilibrium values are ε -maxmin and they are called ε -optimal. A nonzero-sum game may admit equilibrium situations with unequal equilibrium values.

Let us introduce a partial order relation \succ in \mathbb{R}^2 :

$$a \succ b, \text{ if } a_j \geq b_j \text{ (} j = 1, 2),$$

where at least one of the inequalities is strict. Denote by S the set of all situations $s = (x, y)$. The relation of *Pareto preference* on the set S is defined as follows. Let $s^i = (x^i, y^i)$ ($i = 1, 2$). Then

$$s^1 \succ s^2, \text{ if } K^1 \succ K^2, \text{ where } K^i = (K_1(x^i, y^i), K_2(x^i, y^i)), \text{ } i = 1, 2. \quad (3)$$

A situation $s^p = (x_p, y_p)$ is called *Pareto-optimal* if there are no situations $s = (x, y)$ such that $s \succ s^p$.

In an antagonistic game all the situations are Pareto-optimal. A game is called *quasi-antagonistic*, or a *game with opposite interests*, if all the situations of the game are Pareto-optimal, i. e., if for any two situations $(x^1, y^1), (x^2, y^2) \in S$ the following conditions hold:

$$\begin{aligned} K_1(x^1, y^1) < K_1(x^2, y^2) &\iff K_2(x^1, y^1) > K_2(x^2, y^2); \\ K_1(x^1, y^1) = K_1(x^2, y^2) &\iff K_2(x^1, y^1) = K_2(x^2, y^2). \end{aligned} \quad (4)$$

3 Posing the problem

Consider a nonzero-sum game of two players which is a generalization of the antagonistic noisy duel with discrete firing.

The players compete in the conditions of complete information. They have discrete resources $m, n \in \mathbb{N}$ which they use during the time interval

$[0, 1]$. The effectiveness of the j -th player using his resource is described by the function $P_j(t)$ ($j = 1, 2$), which defines the probability of achieving success when using the unit of resource at the moment t . The functions $P_j(t)$ are continuous and increasing, $P_j(0) = 0$, $P_j(1) = 1$, $0 < P_j(t) < 1$ for $t \in (0, 1)$. If one of the players achieves success, the game stops. If a player has used all of his resource and hasn't achieved success, the other player postpones his action until the moment $t = 1$, when his probability of success is equal to one. The profit of the j -th player in the case of his success is equal to A_j , and his loss in the case of his opponent's success is equal to B_j , where

$$A_j \geq 0, \quad B_j \geq 0, \quad A_j + B_j > 0, \quad j = 1, 2. \quad (5)$$

The players' profits are equal to 0 if no one of them achieved success or if the success was achieved by both of them simultaneously. A player's strategy is a function assigning the moment of next action to a pair of amounts of players' current resources. Let us call the described game a *noisy nonzero-sum duel with discrete firing*.

Denote by τ_i ($0 \leq \tau_i \leq \tau_{i-1} \leq 1$, $i = 1, 2, \dots, m$) the moments of time when Player I uses his resource. Analogously, denote by η_i ($0 \leq \eta_i \leq \eta_{i-1} \leq 1$, $i = 1, 2, \dots, n$) the moments of time when Player II uses his resource. The vectors τ and η will be called the *vectors of action moments* of the players. The payoff function $K_j(\tau, \eta)$ is the mathematical expectation of profit received by the j -th player in the case when the players use their resources at the moments of time τ_i, η_j ($i = 1, \dots, m; j = 1, \dots, n$). It is computed in the following way. If $m = 0, n = 0$, then $K_1 = K_2 = 0$. If $m \geq 1, n = 0$, then $K_1 = A_1, K_2 = -B_2$. If $m = 0, n \geq 1$, then $K_1 = -B_1, K_2 = A_2$. Assume that $m \geq 1, n \geq 1$. Set

$$\tau' = (\tau_{m-1}, \dots, \tau_1); \quad \eta' = (\eta_{n-1}, \dots, \eta_1).$$

Then

$$K_1(\tau, \eta) = \begin{cases} A_1 P_1(\tau_m) + (1 - P_1(\tau_m)) K_1(\tau', \eta), & \text{if } \tau_m < \eta_n, \\ A_1 P_1(\tau_m) (1 - P_2(\tau_m)) - B_1 (1 - P_1(\tau_m)) P_2(\tau_m) + \\ + (1 - P_1(\tau_m)) (1 - P_2(\tau_m)) K_1(\tau', \eta'), & \text{if } \tau_m = \eta_n, \\ -B_1 P_2(\eta_n) + (1 - P_2(\eta_n)) K_1(\tau, \eta'), & \text{if } \tau_m > \eta_n. \end{cases} \quad (6)$$

$$K_2(\tau; \eta) = \begin{cases} A_2 P_2(\eta_n) + (1 - P_2(\eta_n)) K_2(\tau, \eta'), & \text{if } \eta_n < \tau_m, \\ A_2 P_2(\tau_m)(1 - P_2(\tau_m)) - B_2(1 - P_1(\tau_m)) P_2(\tau_m) + \\ + (1 - P_1(\tau_m))(1 - P_2(\tau_m)) K_2(\tau', \eta'), & \text{if } \tau_m = \eta_n, \\ -B_2 P_1(\tau_m) + (1 - P_2(\tau_m)) K_2(\tau', \eta), & \text{if } \eta_n > \tau_m. \end{cases} \quad (7)$$

Let us denote $A = (A_1, A_2)$, $B = (B_1, B_2)$ and call A the profit vector and B the loss vector of the players. Introduce the vector-function of effectiveness $P(t) = (P_1(t), P_2(t))$. Let us denote the described game by $\Gamma_{mn}(P, A, B)$ or, for brevity, by Γ_{mn} and its mixed extension by $\bar{\Gamma}_{mn}(P, A, B)$ (or $\bar{\Gamma}_{mn}$).

Suppose that in the duel $\Gamma_{mn}(P, A, B)$ the profit and loss vectors are related by the equation $A_1 = B_2$, $A_2 = B_1$, i. e., the profit of each player is equal to his opponents' loss. Then it follows from the relations (6), (7) that $K_1(\tau, \eta) = -K_2(\tau, \eta)$, i. e., under these conditions the game is antagonistic.

4 Situations of ε -equilibrium

Lemma 1 (Fox, Kimeldorf [3]). *There exists a set $\{t_{ij} \mid i, j \in \mathbb{N}\}$ such that*

$$\prod_{i=1}^m (1 - P_1(t_{in})) + \prod_{j=1}^n (1 - P_2(t_{mj})) = 1, \quad (8)$$

and for all $m, n \in \mathbb{N}$ the following inequalities hold:

$$0 < t_{mn} < \min(t_{m-1,n}, t_{m,n-1}), \text{ where } t_{0n} = t_{m0} = 1.$$

Set

$$\lambda = \min \{1/(A_1 + B_1), 1/(A_2 + B_2)\} / 2. \quad (9)$$

Choose $\varepsilon > 0$ and find δ_j ($j = 1, 2$) such that

$$t_{mn} < \delta_j < \min(t_{m-1,n}, t_{m,n-1}) \text{ and } P_j(\delta_j) < P_j(t_{mn}) + \lambda\varepsilon. \quad (10)$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Let μ, ν be the current values of the players' resources and $\varphi_{\mu\nu}^\varepsilon$ be the uniform distribution concentrated in the interval $[t_{\mu\nu}, t_{\mu\nu} + \delta]$. Define the players' mixed strategies $x^\varepsilon, y^\varepsilon$ in the following way. The strategy x^ε (y^ε) prescribes to choose the next moment of action τ_μ (η_ν) in the random way according to the distribution function $\varphi_{\mu\nu}^\varepsilon$.

Theorem 1. *The situations $(x^\varepsilon, y^\varepsilon)$ in the game $\bar{\Gamma}_{mn}(P, A, B)$ are ε -equilibrium situations. The vector $v_{mn} = (v_{mn}^1, v_{mn}^2)$ defined by the formulas*

$$\begin{aligned} v_{mn}^1 &= A_1 - (A_1 + B_1) \prod_{i=1}^m (1 - P_1(t_{in})) = (A_1 + B_1) \prod_{j=1}^n (1 - P_2(t_{mj})) - B_1; \\ v_{mn}^2 &= (A_2 + B_2) \prod_{i=1}^m (1 - P_1(t_{in})) - B_2 = A_2 - (A_2 + B_2) \prod_{j=1}^n (1 - P_2(t_{mj})) \end{aligned} \quad (11)$$

is the corresponding equilibrium value.

Proof. The second equations in both lines of (11) hold by Lemma 1. Let us obtain recurrence relations for v_{mn} . By (11) we have:

$$\begin{aligned} v_{m-1,n}^1 &= A_1 - (A_1 + B_1) \prod_{i=1}^{m-1} (1 - P_1(t_{in})); \\ v_{mn}^1 &= A_1 - (A_1 + B_1) (1 - P_1(t_{mn})) \prod_{i=1}^{m-1} (1 - P_1(t_{in})) = \\ &= A_1 - (1 - P_1(t_{mn})) (v_{m-1,n}^1 - A_1). \end{aligned}$$

Finally we have an expression for v_{mn}^1 in terms of $v_{m-1,n}^1$:

$$\begin{aligned} v_{0n}^1 &= -B_1 \quad \text{for } n > 0; \\ v_{mn}^1 &= A_1 P_1(t_{mn}) + (1 - P_1(t_{mn})) v_{m-1,n}^1 \quad \text{for } m > 0, n > 0. \end{aligned} \quad (12)$$

Now obtain an expression for v_{mn}^1 in terms of $v_{m,n-1}^1$. By (11) we have:

$$\begin{aligned} v_{m,n-1}^1 &= (A_1 + B_1) \prod_{j=1}^{n-1} (1 - P_2(t_{mj})) - B_1; \\ v_{mn}^1 &= (A_1 + B_1) (1 - P_2(t_{mn})) \prod_{j=1}^{n-1} (1 - P_2(t_{mj})) - B_1 = \\ &= (1 - P_2(t_{mn})) (v_{m,n-1}^1 + B_1) - B_1. \end{aligned}$$

Hence we get:

$$\begin{aligned} v_{m0}^1 &= A_1 \quad \text{for } m > 0; \\ v_{mn}^1 &= -B_1 P_2(t_{mn}) + (1 - P_2(t_{mn})) v_{m,n-1}^1 \quad \text{for } m > 0, n > 0. \end{aligned} \quad (13)$$

Analogously we deduce recurrence relations for v_{mn}^2 in terms of $v_{m-1,n}^2$ and $v_{m,n-1}^2$:

$$\begin{aligned} v_{m0}^2 &= -B_2 \text{ for } m > 0; \\ v_{mn}^2 &= A_2 P_2(t_{mn}) + (1 - P_2(t_{mn})) v_{m,n-1}^2 \text{ for } m > 0, n > 0. \end{aligned} \quad (14)$$

$$\begin{aligned} v_{0n}^2 &= A_2 \text{ for } n > 0; \\ v_{mn}^2 &= -B_2 P_1(t_{mn}) + (1 - P_1(t_{mn})) v_{m-1,n}^2 \text{ for } m > 0, n > 0. \end{aligned} \quad (15)$$

To prove that $(x^\varepsilon, y^\varepsilon)$ are ε -equilibrium situations it is necessary and sufficient to verify the following relations:

$$\overline{K}_1(\tau, y^\varepsilon) \leq \overline{K}_1(x^\varepsilon, y^\varepsilon) + \varepsilon \text{ for any } \tau; \quad (16)$$

$$\overline{K}_2(x^\varepsilon, \eta) \leq \overline{K}_2(x^\varepsilon, y^\varepsilon) + \varepsilon \text{ for any } \eta; \quad (17)$$

$$\lim_{n \rightarrow \infty} \overline{K}_1(x^\varepsilon, y^\varepsilon) = v_{mn}^1; \quad (18)$$

$$\lim_{n \rightarrow \infty} \overline{K}_2(x^\varepsilon, y^\varepsilon) = v_{mn}^2. \quad (19)$$

Lemma 2. *For any $\varepsilon > 0$ there exist such strategies $x^\varepsilon, y^\varepsilon$ that for any pure strategies τ and η of Players I and II the following inequalities hold:*

$$\overline{K}_1(\tau, y^\varepsilon) < v_{mn}^1 + \varepsilon; \quad (20)$$

$$\overline{K}_1(x^\varepsilon, \eta) > v_{mn}^1 - \varepsilon. \quad (21)$$

Proof. For $n = 0$ and arbitrary $m > 0$ by the definition of payoff function we have:

$$K_1 = A_1; \quad K_2 = -B_2.$$

Analogously for $m = 0$ and arbitrary $n > 0$

$$K_1 = -B_1; \quad K_2 = A_2.$$

By (11) we have:

$$v_{m0}^1 = A_1; \quad v_{m0}^2 = -B_2; \quad v_{0n}^1 = -B_1; \quad v_{0n}^2 = A_2.$$

In both cases the inequalities (20), (21) hold.

For arbitrary $m > 0, n > 0$ we proceed by induction on the number of action moments of the players. Assume that Lemma is true for all pairs (μ, ν) for which $\mu \leq m, \nu \leq n, (\mu, \nu) \neq (m, n)$ and prove it for $(\mu, \nu) = (m, n)$.

Every pure strategy of Player I has the following structure. Let $t \in [0, 1]$ denote the planned moment of his first action. If Player II acts and misses at a time $\eta_n < t$, then Player I follows a pure strategy τ^1 in $\Gamma_{m,n-1}$. If Player II does not act before the time t then Player I acts at the time t , and unless Player II also acts at the time t , Player I after that adopts a pure strategy τ^2 in $\Gamma_{m-1,n}$.

Every pure strategy of Player II has the similar structure. Let $u \in [0, 1]$ denote the planned moment of his first action. If Player I acts and misses at a time $\tau_m < u$, then Player II follows a pure strategy η^1 in $\Gamma_{m-1,n}$. If Player I does not act before the time u then Player II acts at the time u , and unless Player I also acts at the time u , Player II after that adopts a pure strategy η^2 in $\Gamma_{m,n-1}$.

The strategy y^ε of Player II is constructed as follows. Fix $\varepsilon > 0$. Choose t randomly according to φ_{mn}^ε . If Player I does not act before the time t , then Player II acts at the time $\eta_n = t$ and, unless Player I also acts at the time t , then adopts the strategy $y^{1,\varepsilon}$ in $\Gamma_{m,n-1}$. If Player I acts and misses at the time $\tau_m < u$, then Player II adopts the strategy $y^{2,\varepsilon}$ in $\Gamma_{m-1,n}$. According to the inductive assumption, we choose $y^{1,\varepsilon}$, $y^{2,\varepsilon}$ such that for any pure strategies τ^1 and τ^2 of Player I the following inequalities hold:

$$\overline{K}_1(\tau^1, y^{1,\varepsilon}) < v_{m,n-1}^1 + \varepsilon/2 \quad (22)$$

$$\overline{K}_1(\tau^2, y^{2,\varepsilon}) < v_{m-1,n}^1 + \varepsilon/2 \quad (23)$$

The strategy x^ε of Player I is constructed similarly. Choose u randomly according to φ_{mn}^ε . If Player II does not act before the time u , then Player I acts at the time $\eta_n = u$ and, unless Player II also acts at the time u , adopts the strategy $x^{1,\varepsilon}$ in $\Gamma_{m-1,n}$. If Player II acts and misses at the time $\tau_m < u$, then Player I adopts the strategy $x^{2,\varepsilon}$ in $\Gamma_{m,n-1}$. According to the inductive assumption, we choose $x^{1,\varepsilon}$, $x^{2,\varepsilon}$ such that for any pure strategies η^1 and η^2 of Player II the following inequalities hold:

$$\overline{K}_1(x^{1,\varepsilon}, \eta^1) > v_{m-1,n}^1 - \varepsilon/2 \quad (24)$$

$$\overline{K}_1(x^{2,\varepsilon}, \eta^2) > v_{m,n-1}^1 - \varepsilon/2 \quad (25)$$

For all strategies described above we have ignored the response to simultaneous actions of players since this event has probability 0 when Player I adopts x^ε or Player II adopts y^ε .

Let us prove the inequality (20).

Note that by the definition of payoff function for any strategies x, y we have:

$$\begin{aligned} -B_j &\leq \overline{K}_j(x, y) \leq A_j \quad (j = 1, 2) \implies \\ \overline{K}_j(x, y) + B_j &\geq 0; \end{aligned} \tag{26}$$

$$A_j - \overline{K}_j(x, y) \geq 0. \tag{27}$$

Let τ be an arbitrary pure strategy of Player I and t be the time of first action in τ . There are three cases to be considered.

1. Suppose $t \in [0, t_{mn}]$. In this case Player I always acts first; so

$$\overline{K}_1(\tau, y^\varepsilon) = A_1 P_1(t) + (1 - P_1(t)) \overline{K}_1(\tau^2, y^{2,\varepsilon}). \tag{28}$$

One can see from the inequality (27) that the right hand side of (28) does not exceed

$$\begin{aligned} (A_1 - \overline{K}_1(\tau^2, y^{2,\varepsilon})) P_1(t_{mn}) + \overline{K}_1(\tau^2, y^{2,\varepsilon}) &= \\ = A_1 P_1(t_{mn}) + (1 - P_1(t_{mn})) \overline{K}_1(\tau^2, y^{2,\varepsilon}). \end{aligned}$$

According to (23), the latter expression is less than

$$A_1 P_1(t_{mn}) + (1 - P_1(t_{mn})) (v_{m-1,n} + \varepsilon/2).$$

Using the recurrence relation (12) we get:

$$A_1 P_1(t_{mn}) + (1 - P_1(t_{mn})) (v_{m-1,n} + \varepsilon/2) = v_{mn} + (1 - P_1(t_{mn})) \varepsilon.$$

Therefore

$$\overline{K}_1(\tau, y^\varepsilon) < v_{mn} + \varepsilon.$$

2. Suppose $t \in (t_{mn}, t_{mn} + \delta)$. In this case either player may act first; so

$$\begin{aligned} \overline{K}_1(\tau, y^\varepsilon) &= \int_{t_{mn}}^t (-B_1 P_2(\xi) + (1 - P_2(\xi)) \overline{K}_1(\tau^1, y^{1,\varepsilon})) d\varphi_{mn}^\varepsilon(\xi) + \\ &+ \int_t^{t_{mn}+\delta} (A_1 P_1(\xi) + (1 - P_1(\xi)) \overline{K}_1(\tau^2, y^{2,\varepsilon})) d\varphi_{mn}^\varepsilon(\xi). \end{aligned}$$

Estimate the first integrand using the relations (26), (22), and (13):

$$\begin{aligned}
& -B_1P_2(\xi) + (1 - P_2(\xi))\overline{K}_1(\tau^1, y^{1,\varepsilon}) = \\
& = \overline{K}_1(\tau^1, y^{1,\varepsilon}) - P_2(\xi)(\overline{K}_1(\tau^1, y^{1,\varepsilon}) + B_1) \leq \\
& \leq \overline{K}_1(\tau^1, y^{1,\varepsilon}) - P_2(t_{mn})(\overline{K}_1(\tau^1, y^{1,\varepsilon}) + B_1) = \\
& -B_1P_2(t_{mn}) + (1 - P_2(t_{mn}))\overline{K}_1(\tau^1, y^{1,\varepsilon}) \leq \\
& \leq -B_1P_2(t_{mn}) + (1 - P_2(t_{mn}))(v_{m,n-1} + \varepsilon/2) = \\
& = v_{mn} + (1 - P_2(t_{mn}))\varepsilon/2 < v_{mn} + \varepsilon.
\end{aligned}$$

Estimate the second integrand using the relations (27), (26), (9), (10), and (12):

$$\begin{aligned}
& A_1P_1(\xi) + (1 - P_1(\xi))\overline{K}_1(\tau^2, y^{2,\varepsilon}) = \\
& = (A_1 - \overline{K}_1(\tau^2, y^{2,\varepsilon}))P_1(\xi) + \overline{K}_1(\tau^2, y^{2,\varepsilon}) \leq \\
& \leq (A_1 - \overline{K}_1(\tau^2, y^{2,\varepsilon}))(P_1(t_{mn}) + \lambda\varepsilon) + \overline{K}_1(\tau^2, y^{2,\varepsilon}) \leq \\
& \leq A_1P_1(t_{mn}) + (1 - P_1(t_{mn}))\overline{K}_1(\tau^2, y^{2,\varepsilon}) + (A_1 + B_1)\lambda\varepsilon \leq \\
& \leq A_1P_1(t_{mn}) + (1 - P_1(t_{mn}))(v_{m-1,n} + \varepsilon/2) + \varepsilon/2 = \\
& = v_{mn} + (1 - P_2(t_{mn}))\varepsilon/2 + \varepsilon/2 < v_{mn} + \varepsilon.
\end{aligned}$$

After integrating we get:

$$\overline{K}_1(\tau, y^\varepsilon) < v_{mn} + \varepsilon.$$

3. Suppose $t \in [t_{mn} + \delta, 1]$. In this case Player II always acts first; so

$$\overline{K}_1(\tau, y^\varepsilon) = \int_{t_{mn}}^{t_{mn}+\delta} (-B_1P_2(\xi) + (1 - P_2(\xi))\overline{K}_1(\tau^1, y^{1,\varepsilon}))d\varphi_{mn}^\varepsilon(\xi).$$

Estimate the integrand using the relations (26), (22), (13):

$$\begin{aligned}
& -B_1P_2(\xi) + (1 - P_2(\xi))\overline{K}_1(\tau^1, y^{1,\varepsilon}) = \\
& = \overline{K}_1(\tau^1, y^{1,\varepsilon}) - (\overline{K}_1(\tau^1, y^{1,\varepsilon}) + B_1)P_2(\xi) \leq \\
& \leq \overline{K}_1(\tau^1, y^{1,\varepsilon}) - (\overline{K}_1(\tau^1, y^{1,\varepsilon}) + B_1)P_2(t_{mn}) = \\
& = -B_1P_2(t_{mn}) + (1 - P_2(t_{mn}))\overline{K}_1(\tau^1, y^{1,\varepsilon}) \leq \\
& \leq -B_1P_2(t_{mn}) + (1 - P_2(t_{mn}))(v_{m,n-1} + \varepsilon/2) < v_{mn} + \varepsilon.
\end{aligned}$$

After integrating we have:

$$\overline{K}_1(\tau, y^\varepsilon) < v_{mn} + \varepsilon.$$

Hence the inequality (20) is proved.

Now let us turn to the inequality (21). Let η be an arbitrary pure strategy of Player II and $u = \eta_n$ be the first action time in η .

There are three cases to be considered.

1. Suppose $u \in [0, t_{mn}]$. In this case Player II acts first; so

$$\overline{K}_1(x^\varepsilon, \eta) = -B_1 P_2(u) + (1 - P_2(u)) \overline{K}_1(x^{2,\varepsilon}, \eta^2).$$

By the relations (26), (22), and (13), we have:

$$\begin{aligned} \overline{K}_1(x^\varepsilon, \eta) &= \overline{K}_1(x^{2,\varepsilon}, \eta^2) - (B_1 + \overline{K}_1(x^{2,\varepsilon}, \eta^2)) P_2(u) \geq \\ &\geq \overline{K}_1(x^{2,\varepsilon}, \eta^2) + (B_1 + \overline{K}_1(x^{2,\varepsilon}, \eta^2)) P_2(t_{mn}) = \\ &= -B_1 P_2(t_{mn}) + (1 - P_2(t_{mn})) \overline{K}_1(x^{2,\varepsilon}, \eta^2) \geq \\ &\geq -B_1 P_2(t_{mn}) + (1 - P_2(t_{mn})) (v_{m,n-1} - \varepsilon/2) = \\ &= v_{mn} - (1 - P_1(t_{mn})) \varepsilon/2 \geq v_{mn} - \varepsilon. \end{aligned}$$

2. Suppose $u \in (t_{mn}, t_{mn} + \delta)$. In this case either player may act first; so

$$\begin{aligned} \overline{K}_1(x^\varepsilon, \eta) &= \int_{t_{mn}}^u (A_1 P_1(\xi) + (1 - P_1(\xi)) \overline{K}_1(x^{2,\varepsilon}, \eta^2)) d\varphi_{mn}^\varepsilon(\xi) + \\ &+ \int_u^{t_{mn}+\delta} (-B_1 P_2(\xi) + (1 - P_2(\xi)) \overline{K}_1(x^{1,\varepsilon}, \eta^1)) d\varphi_{mn}^\varepsilon(\xi). \end{aligned}$$

Estimate the first integrand using the relations (27), (25), and (12):

$$\begin{aligned} A_1 P_1(\xi) + (1 - P_1(\xi)) \overline{K}_1(x^{2,\varepsilon}, \eta^2) &= \\ &= (A_1 - \overline{K}_1(x^{2,\varepsilon}, \eta^2)) P_1(\xi) + \overline{K}_1(x^{2,\varepsilon}, \eta^2) \geq \\ &\geq (A_1 - \overline{K}_1(x^{2,\varepsilon}, \eta^2)) P_1(t_{mn}) + \overline{K}_1(x^{2,\varepsilon}, \eta^2) = \\ &= A_1 P_1(t_{mn}) + (1 - P_1(t_{mn})) \overline{K}_1(x^{2,\varepsilon}, \eta^2) \geq \\ &\geq A_1 P_1(t_{mn}) + (1 - P_1(t_{mn})) (v_{m-1,n} - \varepsilon/2) = \\ &= v_{mn} - (1 - P_2(t_{mn})) \varepsilon > v_{mn} - \varepsilon. \end{aligned}$$

Estimate the second integrand using the relations (26), (27), (9), (10), and (13):

$$\begin{aligned}
& -B_1P_2(\xi) + (1 - P_2(\xi))\overline{K}_1(x^{1,\varepsilon}, \eta^1) = \\
& = \overline{K}_1(x^{1,\varepsilon}, \eta^1) - P_2(\xi)(\overline{K}_1(x^{1,\varepsilon}, \eta^1) + B_1) \geq \\
& \geq \overline{K}_1(\tau^1, y^{1,\varepsilon}) - (P_2(t_{mn}) + \lambda\varepsilon)(\overline{K}_1(x^{1,\varepsilon}, \eta^1) + B_1) \geq \\
& \geq -B_1P_2(t_{mn}) + (1 - P_2(t_{mn}))\overline{K}_1(x^{1,\varepsilon}, \eta^1) - (A_1 + B_1)\lambda\varepsilon \geq \\
& \geq -B_1P_2(t_{mn}) + (1 - P_2(t_{mn}))(v_{m,n-1} - \varepsilon/2) - \varepsilon/2 = \\
& = v_{mn} - (1 - P_2(t_{mn}))\varepsilon/2 - \varepsilon/2 > v_{mn} - \varepsilon.
\end{aligned}$$

After integrating we get:

$$\overline{K}_1(\tau, y^\varepsilon) > v_{mn} - \varepsilon.$$

3. Suppose $u \in [t_{mn} + \delta, 1]$. In this case Player I acts first; so

$$\overline{K}_1(x^\varepsilon, \eta) = \int_{t_{mn}}^{t_{mn}+\delta} (A_1P_1(\xi) + (1 - P_1(\xi))\overline{K}_1(x^{1,\varepsilon}, \eta^1))d\varphi_{mn}^\varepsilon(\xi).$$

By the relations (27), (22), and (12), we have:

$$\begin{aligned}
\overline{K}_1(x^\varepsilon, \eta) & = \int_{t_{mn}}^{t_{mn}+\delta} ((A_1 - \overline{K}_1(x^{1,\varepsilon}, \eta^1))P_1(\xi) + \overline{K}_1(x^{1,\varepsilon}, \eta^1))d\varphi_{mn}^\varepsilon(\xi) \geq \\
& \geq \int_{t_{mn}}^{t_{mn}+\delta} ((A_1 - \overline{K}_1(x^{1,\varepsilon}, \eta^1))P_1(t_{mn}) + \overline{K}_1(x^{1,\varepsilon}, \eta^1))d\varphi_{mn}^\varepsilon(\xi) = \\
& = A_1P_1(t_{mn}) + (1 - P_1(t_{mn}))\overline{K}_1(x^{1,\varepsilon}, \eta^1) \geq \\
& \geq A_1P_1(t_{mn}) + (1 - P_1(t_{mn}))(v_{m-1,n} - \varepsilon/2) = \\
& = v_{mn} - (1 - P_1(t_{mn}))\varepsilon/2 \geq v_{mn} - \varepsilon/2.
\end{aligned}$$

Hence the inequality (21) is proved. \square

Let us continue to prove the theorem. Choose $\varepsilon > 0$. By Lemma 2, we can find a strategy y^ε , satisfying (20) for any pure strategy τ of Player I.

From the inequality (20) it follows that

$$\overline{K}_1(x, y^\varepsilon) < v_{mn}^1 + \varepsilon \quad (29)$$

for any mixed strategy x of Player I. In particular for $x = x^\varepsilon$ we have:

$$\overline{K}_1(x^\varepsilon, y^\varepsilon) < v_{mn}^1 + \varepsilon \implies v_{mn}^1 > \overline{K}_1(x^\varepsilon, y^\varepsilon) - \varepsilon. \quad (30)$$

From the inequality (21) it follows that

$$\overline{K}_1(x^\varepsilon, y) > v_{mn}^1 - \varepsilon \quad (31)$$

for any mixed strategy y of Player II. In particular for $y = y^\varepsilon$ we have:

$$\overline{K}_1(x^\varepsilon, y^\varepsilon) > v_{mn}^1 - \varepsilon \implies v_{mn}^1 < \overline{K}_1(x^\varepsilon, y^\varepsilon) + \varepsilon. \quad (32)$$

Taking in account the relations (29), (32) we obtain

$$\overline{K}_1(x, y^\varepsilon) < v_{mn}^1 + \varepsilon < \overline{K}_1(x^\varepsilon, y^\varepsilon) + \varepsilon$$

for any mixed strategy x of Player I. Hence inequality (16) is proved. The relation (18) follows from (30) and (32).

Due to the symmetry of the setting the relations (17), (19) are proved by the same arguments that (16), (18). \square

Theorem 2. *The maxmin value of the game $\overline{\Gamma}_{mn}(P, A, B)$ coincides with the equilibrium value v_{mn} . The ε -equilibrium strategies $x^\varepsilon, y^\varepsilon$ are ε -maxmin strategies.*

Proof. Let us use the properties of the zero-sum duel $\overline{\Gamma}_{mn}(P, C, \overline{C})$, where $C_1 = A_1, C_2 = B_1, \overline{C} = (C_2, C_1)$. In an antagonistic game the equilibrium value coincides with the maxmin value and ε -equilibrium strategies are ε -maxmin strategies. As the equilibrium value of $\overline{\Gamma}_{mn}(P, C, \overline{C})$ is equal to v_{mn}^1 [4], so the maxmin value of this game is also equal to v_{mn}^1 and the ε -equilibrium strategy x^ε of Player I is his ε -maxmin strategy. Thus the maxmin value of the game $\overline{\Gamma}_{mn}(P, A, B)$ for Player I is equal to v_{mn}^1 , and x^ε is his ε -maxmin strategy.

The statement of the theorem for Player II follows from the symmetry of the setting. \square

5 Pareto-optimal plays

The pair $p = (\tau, \eta)$ of vectors of action moments realized during the game is called a *play*. Let us denote the set of all plays of a noisy duel $\bar{\Gamma}_{mn}(P, A, B)$ by \mathcal{P} . Note that \mathcal{P} is a subset of the set of all situations of the corresponding silent duel (with the same effectiveness functions, resources, and vectors of profit and loss). Namely, \mathcal{P} includes exactly those situations of the silent duel in which after one of the players has used all of his resource, the other one postpones his action until the moment $t = 1$. A play $p^1 \in \mathcal{P}$ is called *Pareto-optimal* if there exist no $p^2 \in \mathcal{P}$ such that $p^2 \succ p^1$. Plays p^1 and p^2 are called *incomparable* if $p^1 \not\succ p^2$ and $p^2 \not\succ p^1$. Plays $p^1 = (\tau^1, \eta^1)$ and $p^2 = (\tau^2, \eta^2)$ are called *equivalent* if $K_j(\tau^1, \eta^1) = K_j(\tau^2, \eta^2)$ ($j = 1, 2$).

A play $p = (\tau, \eta) \in \mathcal{P}$ is called a *T-play* if for any k, l ($1 \leq k \leq m; 1 \leq l \leq n$) at least one of the following two equations holds

$$\tau_k = t_{kl}; \quad \eta_l = t_{kl},$$

where k, l are the current resources of the players.

Lemma 3. *Let $p^T = (\tau^T, \eta^T)$ be an arbitrary T-play with noncoinciding action moments of the players, i. e., $\tau_k \neq \eta_l$ for all k ($1 \leq k \leq m$), l ($1 \leq l \leq n$). Then*

$$K_1(\tau^T, \eta^T) = v_{mn}^1; \quad K_2(\tau^T, \eta^T) = v_{mn}^2. \quad (33)$$

Proof. Let us prove Lemma by induction on the number of action moments of the players. For $n = 0, m > 0$ or $m = 0, n > 0$ the assertion is true as $v_{m0}^1 = A_1; v_{m0}^2 = -B_2, v_{0n}^1 = -B_1; v_{0n}^2 = A_2$. Assume that the statement is true for all pairs (k, l) such that $k \leq m; l \leq n; k + l < m + n$ and prove it for $(k, l) = (m, n)$. Suppose that Player I acts at the moment $t = t_{mn}$. Then using the recursive formula (6) for $\tau_m < \eta_n$ and the inductive assumptions we obtain

$$\begin{aligned} K_1(\tau, \eta) &= A_1 P_1(t_{mn}) - (1 - P_1(t_{mn})) \left(A_1 - (A_1 + B_1) \prod_{i=1}^{m-1} (1 - P_1(t_{in})) \right) = \\ &= A_1 - (A_1 + B_1) \prod_{i=1}^m (1 - P_1(t_{in})) = v_{mn}^1. \end{aligned}$$

Suppose that Player II acts at the moment of time t_{mn} . Then using the formula (6) for $\tau_m > \eta_n$ and the inductive assumptions we obtain

$$\begin{aligned} K_1(\tau, \eta) &= -B_1 P_2(t_{mn}) - (1 - P_2(t_{mn})) \left((A_1 + B_1) \prod_{j=1}^{n-1} (1 - P_2(t_{mj})) - B_1 \right) = \\ &= -B_1 + (A_1 + B_1) \prod_{j=1}^n (1 - P_2(t_{mj})) = v_{mn}^1. \end{aligned}$$

The second player's payoff function is considered in the analogous way. \square

Lemma 4. *Let the plays $p^1 = (\tau^1, \eta^1)$, $p^2 = (\tau^2, \eta^2) \in \mathcal{P}$ satisfy the conditions*

$$\eta_i^1 = \eta_i^2, \quad \tau_i^1 = \tau_i^2 \quad \text{for } i \geq 2; \quad (34)$$

$$\tau_1^1 = t_{11}; \quad \eta_2^1 < t_{11}; \quad \eta_1^1 = 1; \quad \tau_1^2 = t_{11}; \quad \eta_1^2 = t_{11}. \quad (35)$$

Then we have (1) if $A \succ B$, then $p^1 \succ p^2$; (2) if $B \succ A$, then $p^2 \succ p^1$.

Proof. Express the payoff function of the plays p^1 and p^2 in the following way:

$$\begin{aligned} K_1(\tau^1, \eta^1) &= K_1(\tau_m, \dots, \tau_2; \eta_n, \dots, \eta_2) + \prod_{i=2}^m (1 - P_1(\tau_i)) \times \\ &\quad \times \prod_{j=2}^n (1 - P_2(\eta_j)) (A_1 P_1(t_{11}) - B_1 (1 - P_1(t_{11}))); \\ K_1(\tau^2, \eta^2) &= K_1(\tau_m, \dots, \tau_2; \eta_n, \dots, \eta_2) + \prod_{i=2}^m (1 - P_1(\tau_i)) \times \\ &\quad \times \prod_{j=2}^n (1 - P_2(\eta_j)) (A_1 P_1(t_{11}) (1 - P_2(t_{11})) - B_1 (1 - P_1(t_{11})) P_2(t_{11})). \end{aligned}$$

Consider the difference $K_1(\tau^1, \eta^1) - K_1(\tau^2, \eta^2)$. Taking in account that by Lemma 1

$$P_1(t_{11}) + P_2(t_{11}) = 1,$$

we have:

$$\begin{aligned} K_1(\tau^1, \eta^1) - K_1(\tau^2, \eta^2) &= \\ &= (A_1 - B_1)P_1(t_{11})P_2(t_{11}) \prod_{i=2}^m (1 - P_1(\tau_i)) \prod_{j=2}^n (1 - P_2(\eta_j)). \end{aligned} \quad (36)$$

Analogously,

$$\begin{aligned} K_2(\tau^1, \eta^1) - K_2(\tau^2, \eta^2) &= \\ &= (A_2 - B_2)P_1(t_{11})P_2(t_{11}) \prod_{i=2}^m (1 - P_1(\tau_i)) \prod_{j=2}^n (1 - P_2(\eta_j)). \end{aligned} \quad (37)$$

The statement of Lemma follows from (36), (37). \square

A duel may have one of four alternative results: H_0 — no one of players achieves success; H_1 — Player I achieves success; H_2 — Player II achieves success; H_3 — both players achieve success simultaneously. We denote the probability of the result H_i in play (τ, η) by $Q_i(\tau, \eta)$ ($i = 0, 1, 2, 3$). Then:

$$\sum_{i=0}^3 Q_i(\tau, \eta) = 1. \quad (38)$$

Success of both players is possible only if they act simultaneously, because if one of the players achieves success, the game stops. As the payoff function $K_j(\tau; \eta)$ is the mathematical expectation of profit, so

$$K_1(\tau; \eta) = A_1 Q_1(\tau, \eta) - B_1 Q_2(\tau, \eta); \quad (39)$$

$$K_2(\tau; \eta) = A_2 Q_2(\tau, \eta) - B_2 Q_1(\tau, \eta). \quad (40)$$

By (38), we have:

$$K_1(\alpha, \beta) = A_1 - (A_1 + B_1)Q_2(\alpha, \beta) - A_1(Q_0(\alpha, \beta) + Q_3(\alpha, \beta)); \quad (41)$$

$$K_2(\alpha; \beta) = -B_2 + (A_2 + B_2)Q_2(\alpha, \beta) + B_2(Q_0(\alpha, \beta) + Q_3(\alpha, \beta)). \quad (42)$$

Denote by $\mathcal{P}' \subset \mathcal{P}$ the set of plays with noncoinciding action times in which $\tau_1 = 1$ or $\eta_1 = 1$. The complement $\mathcal{P} \setminus \mathcal{P}'$ includes plays in which the players use their last units of resource simultaneously.

Lemma 5. *Let $p' = (\tau', \eta') \in \mathcal{P}'$ and $p = (\tau, \eta) \in \mathcal{P} \setminus \mathcal{P}'$. In this case*
(1) if $p' \succ p$, then $A_1 A_2 > B_1 B_2$; (2) if $p \succ p'$, then $A_1 A_2 < B_1 B_2$.

Proof. Let $(\tau', \eta') \in \mathcal{P}'$, so the relations (41), (42) reduce to the form:

$$K_1(\tau', \eta') = A_1 - (A_1 + B_1)Q_2(\tau', \eta'); \quad (43)$$

$$K_2(\tau', \eta') = -B_2 + (A_2 + B_2)Q_2(\tau', \eta'). \quad (44)$$

Consider the differences $\Delta_j = K_j(\tau, \eta) - K_j(\tau', \eta')$. Applying the formulas (41), (42), (43), (44) we get:

$$\Delta_1 = (A_1 + B_1)I(\tau', \eta'; \tau, \eta) - A_1Q(\tau, \eta); \quad (45)$$

$$\Delta_2 = -(A_2 + B_2)I(\tau', \eta'; \tau, \eta) + B_2Q(\tau, \eta). \quad (46)$$

where

$$\begin{aligned} I(\tau', \eta'; \tau, \eta) &= Q_2(\tau', \eta') - Q_2(\tau, \eta); \\ Q(\tau', \eta') &= Q_0(\tau', \eta') + Q_3(\tau', \eta'). \end{aligned}$$

Let us prove the first assertion of Lemma. If $p' \succ p$, then $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and at least one of this inequalities is strict. Since $A_j + B_j > 0$ ($j = 1, 2$), it follows from (45), (46) that

$$I(\tau', \eta'; \tau, \eta) \leq \frac{A_1}{A_1 + B_1}Q(\tau, \eta); \quad (47)$$

$$I(\tau', \eta'; \tau, \eta) \geq \frac{B_2}{A_2 + B_2}Q(\tau, \eta). \quad (48)$$

Since $Q(\tau, \eta) \neq 0$ for $p = (\tau, \eta) \in \mathcal{P} \setminus \mathcal{P}'$, combining the inequalities (47), (48) and taking in account that at least one of them is strict, we get:

$$\frac{B_2}{A_2 + B_2} < \frac{A_1}{A_1 + B_1} \implies A_1A_2 > B_1B_2.$$

The second statement of the lemma is proved in the analogous way. \square

Lemma 6. *Let $p_1, p_2 \in \mathcal{P}'$. Then p_1 and p_2 are Pareto-incomparable.*

Proof. If $p = (\tau, \eta) \in \mathcal{P}'$, then from the relations (43), (44) it follows that:

$$K_2(\alpha, \beta) = \frac{A_1A_2 - B_1B_2}{A_1 + B_1} - \frac{A_2 + B_2}{A_1 + B_1}K_1(\alpha, \beta). \quad (49)$$

The statement of Lemma immediately follows from (49). \square

Theorem 3. *If in the game $\Gamma_{mn}(P, A, B)$ the coefficients of profit and loss of the players satisfy the condition*

$$A_1 A_2 = B_1 B_2, \quad (50)$$

then the game is quasi-antagonistic.

In the paper [12] a similar theorem about a sufficient condition of quasi-antagonisticity was also proven for continuous and discrete duels.

Proof. Note that if the relation (50) holds, then there exists a number $\lambda > 0$ such that

$$K_1(\tau, \eta) = -\lambda K_2(\tau, \eta). \quad (51)$$

Indeed, suppose $A_1 A_2 > 0$. Then according to (50) $B_1 B_2 > 0$ and $A_1/B_2 = B_1/A_2$. In this case from the formulas (39), (40) one can see that (51) holds with $\lambda = A_1/B_2$. Suppose that one of the numbers A_1 or A_2 is equal to zero. If $A_1 = 0$, then by (5) we have $B_1 \neq 0$, and from (50) it follows $B_2 = 0$, $A_2 \neq 0$. Then in view of the relations (39), (40) we conclude that the equation (51) holds with $\lambda = B_1/A_2$. The case $A_2 = 0$ is considered in the similar way. So it is proved that there exists $\lambda > 0$ for which the equation (51) holds. It follows from (51) that the game is quasi-antagonistic. \square

Theorem 4. *If the coefficients of profit and loss of the players are related by the inequality $A_1 A_2 \geq B_1 B_2$, then T -plays with noncoinciding action moments of the players are Pareto-optimal.*

Proof. If $A_1 A_2 = B_1 B_2$, then by Theorem 3 all plays are Pareto-optimal. In any T -play with noncoinciding action moments the last action of a player happens at $t = 1$, i. e., $\max\{\tau_1, \eta_1\} = 1$. Suppose $A_1 A_2 > B_1 B_2$. We will show that $p \not\succ p'$ for any plays $p \in \mathcal{P}$, $p' \in \mathcal{P}'$. There are two cases to be considered.

1. $p \in \mathcal{P}'$. Then by Lemma 6 $p \not\succ p'$.
2. $p \in \mathcal{P} \setminus \mathcal{P}'$. By Lemma 5, if $p \succ p'$ then $A_1 A_2 < B_1 B_2$ in contradiction to the assumption of the theorem. Therefore $p \not\succ p'$.

\square

Theorem 5. *If $B \succ A$, then T -plays with noncoinciding action moments of the players are not Pareto-optimal.*

Proof. Since according to Lemma 3 all the T -plays with noncoinciding action moments of the players are equivalent, we will give the proof for one T -play only, namely for the play $p^1 = (\tau, \eta)$ in which

$$\begin{aligned}\tau_i &= t_{i1} \text{ for } i = 1, \dots, m; \\ \eta_j &= t_{mj} \text{ for } j = 2, \dots, n; \quad \eta_1 = 1.\end{aligned}$$

Set

$$p^2 = (\tau, \eta^2), \text{ where } \eta^2 = (t_{11}, \eta_2, \dots, \eta_n).$$

The plays p^1, p^2 satisfy the conditions of Lemma 4, therefore

$$B \succ A \implies p^2 \succ p^1.$$

Thus the T -play p^1 and all the other T -plays with noncoinciding action moments of the players are not Pareto-optimal. \square

Theorem 6. *If the duel $\Gamma_{mn}(P, A, B)$ is quasi-antagonistic, then one of the following two conditions holds:*

$$(1) (A_1 - B_1)(A_2 - B_2) < 0; \quad (2) A_j = B_j; \quad j = 1, 2. \quad (52)$$

Proof. Assume that the plays p^1, p^2 satisfy the conditions (34), (35). Suppose that no one of the conditions (52) holds. Then the following two cases are possible.

1. $A \succ B$. Then according to the statement (1) of Lemma 4 one has $p^1 \succ p^2$. Therefore the play p^2 is not Pareto-optimal, and so the game is not quasi-antagonistic.
2. $B \succ A$. Then according to the statement (2) of Lemma 4 one has $p^2 \succ p^1$. Therefore the play p^1 is not Pareto-optimal, and so the game is not quasi-antagonistic.

\square

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